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On the weak-equilibrium condition for derivation of algebraic heat flux model

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ABSTRACT

Analogous to an algebraic Reynolds stress model, the algebraic heat flux model (AHFM) is derived from a second-moment closure by invoking the weak-equilibrium condition. The present study investigates this condition in detail as it applies to the advection and diffusive-transport terms. For the advection term, the correct form of this condition in non-inertial frames is obtained by means of an invariant Euclidean transformation. The validity of the diffusive-transport condition is examined through an *a priori* test using a DNS database for rotating turbulent channel flow with heat transfer. It is shown that the weak-equilibrium condition applied to diffusive-transport term tends to fail in the near-wall region. An alternative form is proposed that is based on an asymptotic analysis of the transport equation budget in the near-wall region. An evaluation of proposed form shows that it has the potential to improve the predictive ability of an ARSM for flows involving system rotation and/or streamline curvature.

1. Introduction

There are currently many studies directed toward developing models for the prediction of heat transfer in turbulent flows. Among the various Reynolds-averaged approaches, it appears that a second-moment closure (SMC) is capable of describing most of the essential features of turbulent flow with heat transfer. However, the SMC is less appealing for application to flows involving complex geometries because of the numerical difficulties associated with its high computation cost compared with a linear eddy viscosity model (EVM). The algebraic heat flux model (AHFM) emerges as an alternative that provides a systematic approach to deriving a non-Boussinesq constitutive relations for turbulent heat flux vector, while retaining a many of the features that a SMC possesses. Thus, the interest in deriving an algebraic vector heat flux representation for predicting turbulent heat transfer has increased.

Numerous studies have been devoted to the development of complex algebraic representations for the turbulent heat flux (Rogers et al., 1989; Lai and So, 1990; So and Sommer, 1996; Dol et al., 1997; Shabany and Durbin, 1997; Rokni, 2000; Abe and Suga, 2001; Hattori et al., 2006). Wikström et al. (2000) utilized a systematic modeling approach for forming an implicit algebraic relation for the turbulent heat flux and proposed a method to obtain a fully explicit form from this using the Cayley–Hamilton theorem for two- and three-dimensional flows. So et al. (2004) presented a method for deriving an explicit algebraic model for two-dimensional incompressible non-isothermal turbulent flows with the aid of tensor representation theory. While all these studies have contributed to the development of algebraic representations for the heat flux vector, they were inherently limited to flows in inertial frames. Consequently, the proper general form of such an algebraic representation for the turbulent heat flux vector in the non-inertial frames has been left unexplored.

A general algebraic relation is first obtained from the differential transport equation for the turbulent heat flux by invoking the weak-equilibrium condition. Analogous to the derivation of algebraic Reynolds stress model (ARSM), this weak-equilibrium condition assumes that the advection of the turbulent normalized heat flux is zero. Since the original advection assumption is only valid for inertial frames, it is necessary to identify the proper form for non-inertial frames. The same issue was encountered in the derivation of algebraic representations for the Reynolds stress anisotropy tensor and has been resolved by invoking a frame-invariant transformation to account for rotation and curvature effects correctly (Speziale, 1979, 1998; Girimaji, 1997; Weis and Hutter, 2003; Gatski and Wallin, 2004; Gatski, 2004; Hamba, 2006). As pointed out by Hamba (2006), the frame-invariant property is not only a kinematic requirement on the mathematical formulation, it also serves as a highly useful constraint and tool to form constitutive equations (Speziale, 1998; Hamba, 2006). By invoking the frame-invariant concept, the resultant constitutive equations for the Reynolds stress anisotropies are independent of the reference frames, whether inertial or non-inertial. It has been demonstrated that the resultant frame-invariant algebraic model for the Reynolds stress anisotropy is capable of properly accounting for system rotation and streamline curvature effects (Jongen et al., 1998a,b; Gatski and Wallin, 2004). Fortunately, it is straightforward to extend this

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methodology to the modeling of vector heat fluxes in non-inertial frames. By applying this frame-invariant constraint, a coordinate frame-free algebraic heat flux model can be derived, that can account for system rotation and streamline curvature effects. Nagano and Hattori (2003) evaluated an explicit expression for the turbulent heat flux; however, it appears that the algebraic formulation employed in that is not frame-invariant. Another part of the weak-equilibrium condition assumes that the diffusion and transport term in the budget of the turbulent normalized heat flux equation is negligible. This condition is the diffusive-transport constraint. This constraint removes the differential terms that are associated with the diffusion and transport processes. Obviously, its validity is crucial for constructing an accurate AHFM. Wikström et al. (2000) investigated the validity of the diffusive-transport constraint by comparing the magnitudes of individual terms in the transport equation for normalized heat flux. They concluded that the diffusive-transport constraint was appropriate for the streamwise component except near the wall. For the wall-normal component, this constraint was not well supported even in the center of the channel. Nevertheless, their work was limited to inertial frames, and the validity of this constraint in the non-inertial frames was not assessed. In order to apply the AHFM to flows involving rotation and curvature effects this constraint needs to be assessed further. Following (Wikström et al., 2000), an assessment of the diffusive-transport constraint in non-inertial frames can be done by an analysis of the budget of the normalized heat flux equation using DNS data. Qiu et al. (2008) proposed a near-wall correction for the diffusive-transport constraint applied to an ARSM based on a budget analysis of the transport equation for the Reynolds stress anisotropy. The current study will extend this work to the modification of the diffusive-transport constraint for an AHFM.

The present study investigates the validity and subsequent modification of current weak-equilibrium conditions applicable to vector heat flux transport equations in non-inertial frames. The proper form of the advection assumption will be derived by invoking a frame-invariant property to account for the rotation and curvature effects correctly. In addition, it will also be shown that the transport equation for turbulent heat flux can be written in Euclidean-invariant form by introducing the Jaumann–Noll derivative. The diffusive-transport constraint will also be addressed in detail to show that it is not suitable for flows in non-inertial frames. Based on a budget analysis, a proposal is made for a near-wall correction to the current diffusive-transport constraint. An *a priori* test of the near-wall correction will be performed for the rotating channel flow with heat transfer using DNS data.

2. Algebraic model for turbulent heat flux

In this section, a brief description of the derivation of the algebraic heat flux model from the differential transport equation is given (Wikström et al., 2000; So et al., 2004). The weak-equilibrium condition is invoked to obtain the algebraic relations for the turbulent heat flux analogous to the derivation of algebraic Reynolds stress models.

The Reynolds-averaged equation for the turbulent mean flow (without buoyancy) can be written as

$$\frac{\partial U_i}{\partial x_i} = 0, \tag{1a}$$

$$\frac{DU_i}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial U_i}{\partial x_j} - \overline{u_i u_j}\right), \tag{1b}$$

$$\frac{D\Theta}{Dt} = \frac{\partial}{\partial x_j}\left(\alpha\frac{\partial \Theta}{\partial x_j} - \overline{u_j\theta}\right), \tag{1c}$$

where U_i is the mean velocity, P is the mean pressure, ρ and ν are the constant density and kinematic viscosity, respectively, $\overline{u_i u_j}$ is the Reynolds stress, α is the thermal diffusivity, Θ is the mean temperature, and $\overline{u_i\theta}$ is the turbulent heat flux with θ being the temperature fluctuation.

Analogous to the transport equation for Reynolds stress, the exact transport equation for the turbulent heat flux in an inertial frames (without buoyancy) can be written as

$$\frac{D\overline{u_i\theta}}{Dt} = P_{i\theta} + \phi_{i\theta} + \mathscr{D}_{i\theta} - \varepsilon_{i\theta}, \tag{2}$$

where $P_{i\theta}$ is the production due to the mean temperature and velocity gradient, $\phi_{i\theta}$ is the pressure temperature-gradient correlation term (also known as P&T-Corr.), $\mathscr{D}_{i\theta}$ is the combination of the viscous diffusion, turbulent transport and pressure transport, and $\varepsilon_{i\theta}$ is the dissipation term (Dol et al., 1997). These terms are given by

$$P_{i\theta} = -\overline{u_i u_j}\frac{\partial \Theta}{\partial x_j} - \overline{u_j\theta}\frac{\partial U_i}{\partial x_j}, \tag{3a}$$

$$\phi_{i\theta} = \overline{\frac{p}{\rho}\frac{\partial \theta}{\partial x_i}}, \tag{3b}$$

$$\mathscr{D}_{i\theta} = \frac{\partial}{\partial x_j}\left(\alpha\overline{\frac{\partial\theta}{\partial x_j}u_i} + \nu\overline{\theta\frac{\partial u_i}{\partial x_j}}\right) - \frac{\partial\overline{\theta u_i u_j}}{\partial x_j} - \frac{1}{\rho}\frac{\partial\overline{\theta p}}{\partial x_i}, \tag{3c}$$

$$\varepsilon_{i\theta} = (\alpha+\nu)\overline{\frac{\partial\theta}{\partial x_j}\frac{\partial u_i}{\partial x_j}}. \tag{3d}$$

It is noted that the terms containing correlations of second or higher order in Eq. (2) need to be modeled, and $P_{i\theta}$ can be treated in an exact manner using second- or lower-order variables. Generally, a transport model for the turbulent heat flux can be expressed in terms of the mean velocity and temperature-gradient, the Reynolds stresses and heat fluxes, and corresponding time scales.

In order to obtain an algebraic relation for normalized turbulent heat flux, the turbulent kinetic energy $k(=\overline{u_i u_i}/2)$ and the temperature variance $k_\theta(=\overline{\theta^2}/2)$ are also necessary. Their transport equations can be written as

$$\frac{Dk}{Dt} = \mathscr{D}_k + P_k - \varepsilon_k, \tag{4a}$$

$$\frac{Dk_\theta}{Dt} = \mathscr{D}_\theta + P_\theta - \varepsilon_\theta. \tag{4b}$$

In analogy with the derivation of the transport equation for Reynolds stress anisotropy $b_{ij}(=\overline{u_i u_j}/2k - \delta_{ij}/3)$ (Gatski and Wallin, 2004), one can obtain the transport equation for turbulent normalized heat flux $\xi_i(=\overline{u_i\theta}/(k^{1/2}k_\theta^{1/2}))$ (Wikström et al., 2000; So et al., 2004; Hattori et al., 2006)

$$\frac{D\xi_i}{Dt} = \frac{1}{k^{1/2}k_\theta^{1/2}}(P_{i\theta} + \phi_{i\theta} - \varepsilon_{i\theta}) - \frac{\xi_i}{2}\left[\tau_k\left(\frac{P_k}{\varepsilon_k} - 1\right) + \tau_\theta\left(\frac{P_\theta}{\varepsilon_\theta} - 1\right)\right] + \mathscr{D}_i^a, \tag{5}$$

where $\tau_k = \varepsilon_k/k$ and $\tau_\theta = \varepsilon_\theta/k_\theta$ are time scales. Note that the concept of normalized heat flux itself directly conflicts with the linearity principle proposed by Pope (1983), but has been often abandoned in many scalar transport models (Wikström et al., 2000; So et al., 2004; Hattori et al., 2006). The term $\mathscr{D}_i^a$ is the diffusion and transport of ξ_i, which is

$$\mathscr{D}_i^a = \frac{D_{i\theta}}{k^{1/2}k_\theta^{1/2}} - \frac{1}{2}\xi_i\left(\frac{\mathscr{D}_\theta}{k_\theta} + \frac{\mathscr{D}_k}{k}\right). \tag{6}$$

Eq. (5) is the transport equation for ξ_i, and which is the basis for the derivation of the algebraic model. By invoking the weak-equilibrium condition (Wikström et al., 2000; So et al., 2004), i.e.,

$$\frac{D\xi_i}{Dt} = 0, \tag{7a}$$

$$\mathscr{D}_i^a = 0, \tag{7b}$$

one can reduce Eq. (5) to an approximate form for the turbulent normalized heat flux ξ_i:

$$0 = \frac{1}{k^{1/2}k_\theta^{1/2}}(P_{i\theta} + \phi_{i\theta} - \varepsilon_{i\theta}) - \frac{\xi_i}{2}\left[\tau_k\left(\frac{P_k}{\varepsilon_k} - 1\right) + \tau_\theta\left(\frac{P_\theta}{\varepsilon_\theta} - 1\right)\right]. \tag{8}$$

To extract the AHFM, specific models for the pressure temperature-gradient correlation $\phi_{i\theta}$ and the dissipation term $\varepsilon_{i\theta}$ are necessary. A rather general model for the combined effect of $\phi_{i\theta}$ and $\varepsilon_{i\theta}$ that has been studied previously (Wikström et al., 2000) is considered here, and can be written as

$$\phi_{i\theta} - \varepsilon_{i\theta} = -\left(C_{1\theta} + C_{5\theta}\frac{k}{\varepsilon k_\theta}\overline{u_{j\theta}}\frac{\partial\Theta}{\partial x_j}\right)\frac{\varepsilon}{k}\overline{u_{i\theta}} + C_{2\theta}\overline{u_{j\theta}}\frac{\partial U_i}{\partial x_j} + C_{3\theta}\overline{u_{j\theta}}\frac{\partial U_j}{\partial x_i} + C_{4\theta}\overline{u_i u_j}\frac{\partial\Theta}{\partial x_j}, \tag{9}$$

where $C_{1\theta} \sim C_{5\theta}$ are model coefficients. Substituting Eq. (9) into Eq. (8), one obtains

$$0 = -C_b\left(2b_{ij} + \frac{2\delta_{ij}}{3}\right)\Theta_j - C_S S_{ij}\xi_j - C_W W_{ij}\xi_j - \frac{\xi_i}{2}\left\{\tau_k\left(\frac{P_k}{\varepsilon_k} - 1 + 2C_{1\theta}\right) + \tau_\theta\left[\frac{P_\theta}{\varepsilon_\theta}(1 - 2C_{5\theta}) - 1\right]\right\}, \tag{10}$$

where $C_b = 1 - C_{4\theta}$, $C_S = 1 - C_{2\theta} - C_{3\theta}$, $C_W = 1 - C_{2\theta} + C_{3\theta}$ and $\Theta_i = (k/k_\theta)^{1/2}(\partial\Theta/\partial x_i)$. The strain-rate tensor S_{ij} and vorticity tensor W_{ij} are given as

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \tag{11a}$$

$$W_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right). \tag{11b}$$

Eq. (10) can be expressed in a functional form as

$$0 = f_i(b_{km}, S_{km}, W_{km}, \xi_m, \Theta_m). \tag{12}$$

Eq. (10) is the general model equation used for predicting the turbulent heat flux; however it is only valid for flows in inertial frames. In non-inertial frames, modification must be made to Eq. (10) to account for system rotation and/or streamline curvature effects. Since the weak-equilibrium condition is the basis for deriving the algebraic representation from the modeled normalized heat flux equation, it is straightforward to identify an extended weak-equilibrium condition that is valid both in inertial and non-inertial frames.

3. Frame-invariant form of AHFM

3.1. Frame-invariant form of transport equation for ξ_i

The transformation between inertial and non-inertial frames for the normalized heat flux transport equation is briefly described here. Following the work of Gatski and Wallin (2004) and Hamba (2006), the rectangular coordinates x_i^* in the non-inertial frame transforms to the coordinates in the inertial frame x_i as

$$x_i = Q_{ij}x_j^*, \tag{13}$$

where Q_{ij} is an orthogonal transformation tensor.

The system rotation tensor expressed in the x_i^* coordinates is given by

$$\Omega_{ij}^* = \frac{dQ_{ki}}{dt}Q_{kj} = \epsilon_{jik}\omega_k^*, \tag{14}$$

where ϵ_{ijk} is the permutation tensor, and ω_k^* is the angular rotation rate vector. The system rotation tensor expressed in the x_i coordinates Ω_{ij} is zero by definition.

Under the above transformation rule, the variables appearing in the transport equation of the normalized turbulent heat flux can be transformed as

$$b_{ij} = Q_{ik}b_{km}^*Q_{mj}^{\mathrm{T}}, \tag{15a}$$

$$S_{ij} = Q_{ik}S_{km}^*Q_{mj}^{\mathrm{T}}, \tag{15b}$$

$$W_{ij} = Q_{ik}(W_{km}^* + \Omega_{km}^*)Q_{mj}^{\mathrm{T}}, \tag{15c}$$

$$\xi_i = Q_{ij}\xi_j^*. \tag{15d}$$

For a Euclidean transformation, it is readily seen that, b_{ij}, S_{ij} and ξ_i are all frame-invariant while the vorticity tensor W_{ij} is not. However, W_{ij} can be made frame-invariant by adding a measure of the non-inertial frame rotation rate Ω_{ij}^*

$$\overline{W}_{ij}^* = W_{ij}^* + \Omega_{ij}^*. \tag{16}$$

Similarly, the material derivative of the normalized turbulent heat flux $\mathrm{D}\xi_j/\mathrm{D}t$ can be transformed as

$$\frac{\mathrm{D}\xi_j}{\mathrm{D}t} = Q_{ji}\left(\frac{\mathrm{D}\xi_i^*}{\mathrm{D}t} + \Omega_{ik}^*\xi_k^*\right). \tag{17}$$

The transport equation of turbulent normalized heat flux now can be transformed to the coordinates x_i^*, which can be written as

$$\frac{\mathrm{D}\xi_i^*}{\mathrm{D}t} + \Omega_{ik}^*\xi_k^* = f_i^*(b_{km}^*, S_{km}^*, \overline{W}_{km}^*, \xi_m^*, \Theta_m^*). \tag{18}$$

The above result shows that the normalized turbulent heat flux equation, given by Eq. (18), is not frame-invariant with respect to a change of coordinate system under a Euclidean transformation, since Eq. (17) is not frame-invariant. This is, however, inconsistent with the general understanding that physical laws should be independent of the choice of coordinate systems. This apparent inconsistency can be overcome by introducing the Jaumann–Noll derivative (Trusov, 1987) also called corotational derivative (Thiffeault, 2001).

$$\frac{\overline{\mathrm{D}}\mathbf{a}}{\mathrm{D}t} = \frac{\mathrm{D}\mathbf{a}}{\mathrm{D}t} + \mathbf{\Omega a}, \tag{19a}$$

$$\frac{\overline{\mathrm{D}}\mathbf{b}}{\mathrm{D}t} = \frac{\mathrm{D}\mathbf{b}}{\mathrm{D}t} + \mathbf{b\Omega} - \mathbf{\Omega b}, \tag{19b}$$

with $\mathbf{a}$ and $\mathbf{b}$ being a vector and tensor, respectively. Applying Eq. (19a) to Eq. (17), one can derive a frame-invariant form of the material derivative of the turbulent normalized heat flux

$$\frac{\overline{\mathrm{D}}\xi_j^*}{\mathrm{D}t} = \frac{\mathrm{D}\xi_j^*}{\mathrm{D}t} + \Omega_{jk}^*\xi_k^*. \tag{20}$$

Eq. (18) then becomes

$$\frac{\overline{\mathrm{D}}\xi_j^*}{\mathrm{D}t} = f_j^*(b_{km}^*, S_{km}^*, \overline{W}_{km}^*, \xi_m^*, \Theta_m^*). \tag{21}$$

In this form, the transport equation of normalized turbulent heat flux, Eq. (21), is now frame-invariant, since the corotational derivative $\overline{\mathrm{D}}\xi_j^*/\mathrm{D}t$ is a frame-invariant operator since $\overline{W}_{ij}^*$ is frame-invariant, and the RHS of Eq. (21) is expressed in terms of frame-invariant variables. Consequently, any model expression derived from Eq. (21) should also be frame-invariant.

3.2. Euclidean invariant form of AHFM

Eq. (21) is not suitable for curved flows, and should be further generalized. To this end, the coordinate system $x_i^\dagger$ that is embedded in the flow and the coordinate system x_i^* in which the observer is fixed are considered here. The transformation rule between $x_i^\dagger$ system and inertial system is given by

$$x_i^\dagger = T_{ij}x_j, \tag{22}$$

where T_{ij} is a proper orthogonal transformation tensor.

With this transformation rule, the turbulent normalized heat flux equation can be described in the $x_i^\dagger$ system as

$$\frac{D\xi_i^\dagger}{Dt} + \Omega_{ik}^{(r)\dagger}\xi_k^\dagger = f_i^\dagger\left(b_{km}^\dagger, S_{km}^\dagger, W_{km}^\dagger + \Omega_{km}^{(r)\dagger}, \xi_m^\dagger, \Theta_m^\dagger\right), \quad (23)$$

where $\Omega_{ij}^{(r)\dagger} = T_{ik}\mathrm{d}T_{kj}^{\mathrm{T}}/\mathrm{d}t$ is the rotation rate of the $x_i^\dagger$ system expressed in the $x_i^\dagger$ system. Eq. (23) can be written in the inertial system as

$$\frac{D\xi_i^\dagger}{Dt} + \Omega_{ik}^{(r)}\xi_k = f_i(b_{km}, S_{km}, W_{km}, \xi_m, \Theta_m), \quad (24)$$

where $\Omega_{ij}^{(r)} = T_{ik}^{\mathrm{T}}\Omega_{km}^{(r)\dagger}T_{mj}$ is the rotation rate of the $x_i^\dagger$ system expressed in the inertial system. By applying the weak-equilibrium condition $D\xi_i^\dagger/Dt = 0$, the resultant implicit algebraic equation for ξ_i will have the following form in the inertial system

$$f_i(b_{km}, S_{km}, W_{km}, \xi_m, \Theta_m) - \Omega_{ik}^{(r)}\xi_k = 0. \quad (25)$$

Once again, considering the observer in the x_i^* coordinate system, it is straightforward to transform Eq. (24) to the non-inertial system x_i^*. It follows

$$Q_{ji}T_{ji}^{\mathrm{T}}\frac{D\xi_j^\dagger}{Dt} + \Omega_{ik}^{(r)*}\xi_k^* = f_i^*\left(b_{km}^*, S_{km}^*, W_{km}^* + \Omega_{km}^{(r)*}, \xi_m^*, \Theta_m^*\right). \quad (26)$$

Irrespective of the coordinate system, the correct form of the weak-equilibrium condition should be (Gatski and Wallin, 2004)

$$D\xi_j^\dagger/Dt = 0, \quad (27)$$

which is an extension of the original condition. The resultant implicit algebraic equation for ξ_j in the non-inertial system can then be given by

$$\Omega_{ik}^{(r)*}\xi_k^* = f_i^*\left(b_{km}^*, S_{km}^*, W_{km}^* + \Omega_{km}^{(r)*}, \xi_m^*, \Theta_m^*\right). \quad (28)$$

It is clear that the AHFM written in Eq. (28) is frame-invariant, since it is expressed in terms of frame-invariant variables. It is important to note that Ω_{ij}^* is different from $\Omega_{ij}^{(r)*}$. The former represents a measure of the rotation rate of the flow, while the latter represents the rotation rate of the observer. If the x_i^* system coincides with the $x_i^\dagger$ system, $\Omega_{ij}^* = \Omega_{ij}^{(r)*}$ is obtained. Note that $\Omega_{ij}^{(r)*}$ should be used for general cases, for example, in the case of curved turbulent flows, which is usually analyzed relative to an observer fixed in an inertial frame. Consequently, the problem that arises is how to measure $\Omega_{ij}^{(r)*}$ for such curved flows. There are some works related to this issue, such as Girimaji (1997), Gatski and Jongen (2000), Wallin and Johansson (2002), among others.

Eq. (10), which is expressed in an inertial frame, can be rewritten in a non-inertial frames as

$$\Omega_{ij}^{(r)*}\xi_j^* = -C_b\left(2b_{ij}^* + \frac{2\delta_{ij}}{3}\right)\Theta_j^* - C_S S_{ij}^*\xi_j^* - C_W\left(W_{ij}^* + \Omega_{ij}^{(r)*}\right)\xi_j^*$$
$$- \frac{\xi_i^*}{2}\left\{\tau\left(\frac{P_k}{\varepsilon_k} - 1 + 2C_{1\theta}\right) + \tau_\theta\left[\frac{P_\theta}{\varepsilon_\theta}(1 - 2C_{5\theta}) - 1\right]\right\}. \quad (29)$$

Eq. (29), which is derived based on an extended weak-equilibrium condition (Eq. (27)), has the ability to predict the turbulent normalized heat flux for flows in the non-inertial frames. By comparing with Eq. (10), one can see that Eq. (29) has an extra term $\Omega_{ij}^{(r)*}\xi_j^*$ that describes the advection of ξ_i in the non-inertial frame, and the mean vorticity in Eq. (10) has been replaced with the absolute vorticity.

3.3. A priori test of extended advection assumption

To demonstrate the validity of the extended advection assumption, an *a priori* test is performed using a DNS database (Nishimura and Kasagi, 1996; Kasagi and Iida, 1999; Elsamni and Kasagi, 2001). The test case adopted here is a fully developed turbulent flow in a plane channel that is rotated at a specified angular velocity around its spanwise axis. The Coriolis force arising from the imposed system rotation enhances the turbulence along the pressure side, while reduces the turbulent activity along the suction side. The two walls are assumed to be kept at different, but constant temperatures, and any buoyancy effect is neglected. The simulated channel flow is characterized by a bulk Reynolds number, Re_b (based on the channel half-width h and the bulk velocity U_b), of 4750, and a Prandtl number, Pr, of 0.71. The rotation number is defined as

$$Ro = 2\omega_m h/U_b, \quad (30)$$

where ω_m is the system angular velocity.

Eq. (5) can be transformed to a non-inertial frame and rewritten as

$$\frac{D\xi_i^*}{Dt} + \Omega_{ij}^*\xi_j^* = -\left(2b_{ij}^* + \frac{2\delta_{ij}}{3}\right)\Theta_j^* - S_{ij}^*\xi_j^* - \left(W_{ij}^* + \Omega_{ij}^*\right)\xi_j^*$$
$$+ \frac{1}{k^{1/2}k_\theta^{1/2}}(\phi_{i\theta} - \varepsilon_{i\theta}) - \frac{\xi_i^*}{2}\left[\tau_k\left(\frac{P_k}{\varepsilon_k} - 1\right) + \tau_\theta\left(\frac{P_\theta}{\varepsilon_\theta} - 1\right)\right] + \mathscr{D}_i^a. \quad (31)$$

If the original advection assumption (the left-hand side of Eq. (31) is set to zero), one has

$$0 = -\left(2b_{ij}^* + \frac{2\delta_{ij}}{3}\right)\Theta_j^* - S_{ij}^*\xi_j^* - \left(W_{ij}^* + \Omega_{ij}^*\right)\xi_j^* + \frac{1}{k^{1/2}k_\theta^{1/2}}(\phi_{i\theta} - \varepsilon_{i\theta})$$
$$- \frac{\xi_i^*}{2}\left[\tau_k\left(\frac{P_k}{\varepsilon_k} - 1\right) + \tau_\theta\left(\frac{P_\theta}{\varepsilon_\theta} - 1\right)\right] + \mathscr{D}_i^a, \quad (32)$$

and if the extended assumption Eq. (27) is applied, one has

$$\Omega_{ij}^*\xi_j^* = -\left(2b_{ij}^* + \frac{2\delta_{ij}}{3}\right)\Theta_j^* - S_{ij}^*\xi_j^* - \left(W_{ij}^* + \Omega_{ij}^*\right)\xi_j^* + \frac{1}{k^{1/2}k_\theta^{1/2}}(\phi_{i\theta} - \varepsilon_{i\theta})$$
$$- \frac{\xi_i^*}{2}\left[\tau_k\left(\frac{P_k}{\varepsilon_k} - 1\right) + \tau_\theta\left(\frac{P_\theta}{\varepsilon_\theta} - 1\right)\right] + \mathscr{D}_i^a. \quad (33)$$

If models for the pressure temperature-gradient correlation $\phi_{i\theta}$, the dissipation term $\varepsilon_{i\theta}$, and diffusive-transport term $\mathscr{D}_i^a$, are specified, Eqs. (32) and (33) are easily cast into an implicit form for the AHFM (c.f. Eq. (29)). However, in order to validate the extended form of the advection assumption, no model influence should be allowed; therefore, Eqs. (32) and (33) are used to perform the *a priori* test directly. The DNS data is used to obtain $\phi_{i\theta}$, $\varepsilon_{i\theta}$ and $\mathscr{D}_i^a$ so that the residuals of Eqs. (32) and (33) can be computed.

Since no models are introduced, the magnitude of any residual can be directly associated with the validity of the two assumptions. The results shown in Fig. 1 are the distribution for $Ro = 0.159$. It is shown that the extended assumption gives practically zero residuals for the two components ξ_1 and ξ_2 across the channel. This means that the extended assumption is able to fully account for the rotation effect for flows in non-inertial frames. This is in contrast to the original assumption where large residuals across the channel for all two components are shown.

4. The diffusive-transport constraint

The weak-equilibrium condition consists of an advection assumption and a diffusive-transport constraint. The previous section focused on the advection assumption, which was adapted to the non-inertial frames. In this section, the focus will be on the diffusive-transport constraint. It is true that the system rotation and streamline curvature can significantly influence the transport process in turbulent flows. This can be observed from numerous computations and experimental studies, such as Kasagi et al. (1992), Matsubara and Alfredsson (1996), Yamawaki et al. (2002), Wu

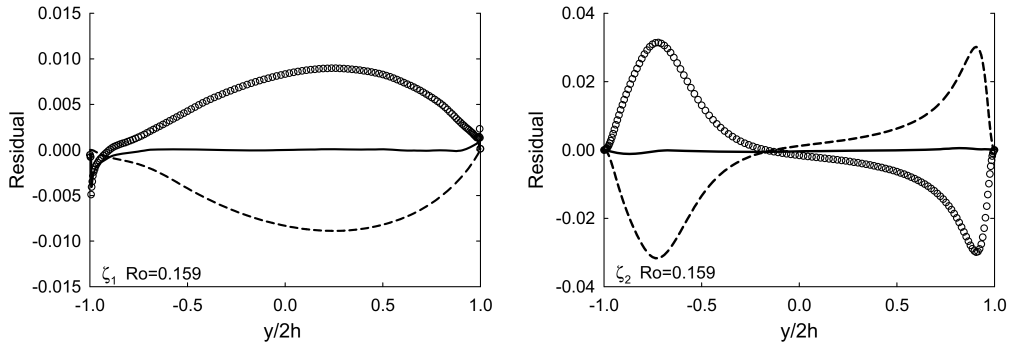

Fig. 1. The *a priori* test of extended advection assumption. Residual of Eq. (32) (circle) compared to that of Eq. (33) (solid line). The dash line is for the extra term $\Omega_{ij}\xi_j$.

and Kasagi (2004), Liu and Lu (2007). Consequently, a question arises whether the usual assumption for diffusive- transport can hold in flows involving rotation and curvature effects. The same question for the diffusive- transport assumption associated with Reynolds stress anisotropy tensor has been recently explored by Qiu et al. (2008) using a budget analysis of the Reynolds stress anisotropy equation together with the near-wall asymptotic behavior. For the current study, an analogous strategy will be employed to address the issue of the diffusive-transport assumption associated with turbulent normalized heat flux.

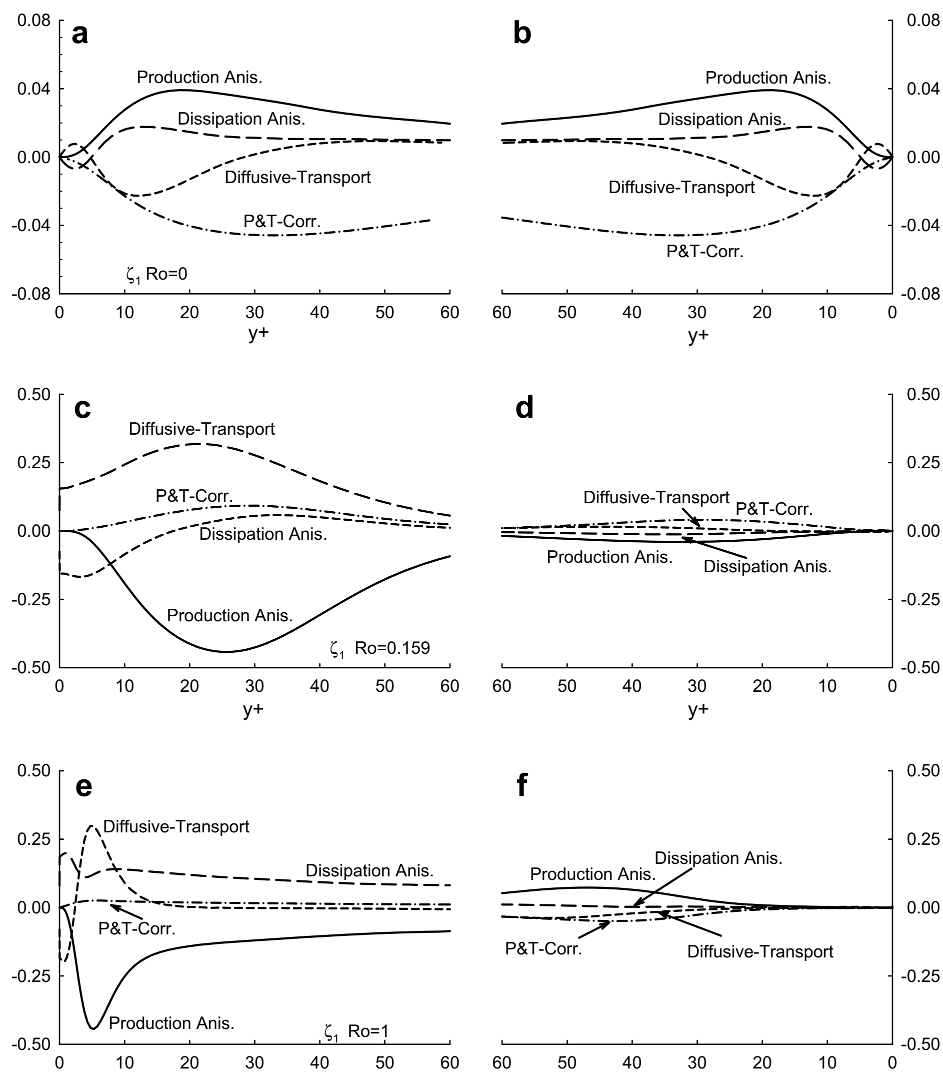

Fig. 2. The budget of Eq. (34) for ξ_1 LHS: pressure side, RHS: suction side.

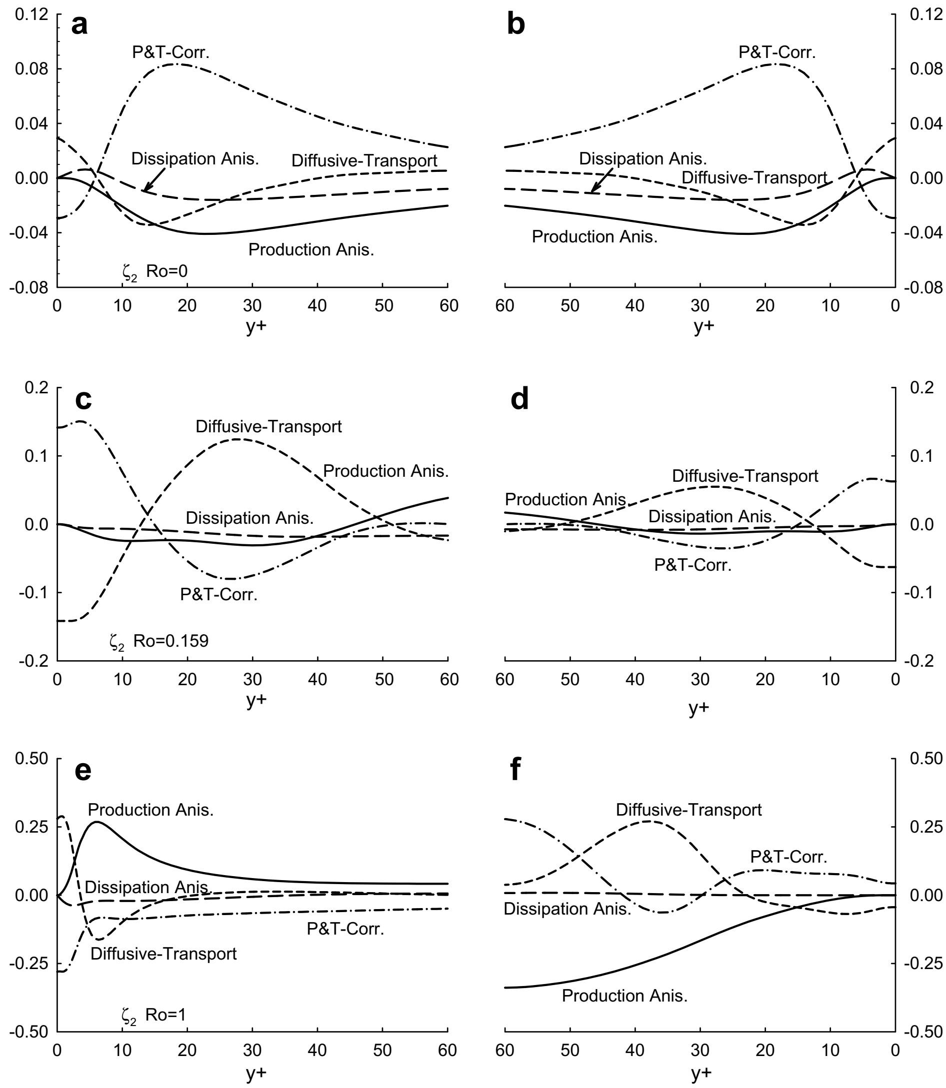

Fig. 3. The budget of Eq. (34) for ξ_2 LHS: pressure side, RHS: suction side.

Table 1
Near-wall behavior of budget terms in Eq. (34)

i	1	2
Production	$\mathcal{O}(y^3)$	$\mathcal{O}(y^3)$
Diffusive-transport	$\mathcal{O}(y)$	$-(1/\rho)\overline{b_\theta a_p}$
P&T-Corr.	$\mathcal{O}(y^2)$	$(1/\rho)\overline{b_\theta a_p}$
Dissipation	$\mathcal{O}(y)$	$\mathcal{O}(y)$

4.1. Budget of turbulent normalized heat flux equation

Analogous to the derivation of the transport equation for Reynolds stress anisotropy b_{ij}, one can obtain a transport equation for the normalized turbulent heat flux ξ_i in fully developed rotating channel flow

$$0 = \left[P_{i\theta} - \frac{\xi_i}{2}\left(RP_k + \frac{P_\theta}{R}\right)\right] + \left[\mathscr{D}_{i\theta} - \frac{\xi_i}{2}\left(R\mathscr{D}_k + \frac{\mathscr{D}_\theta}{R}\right)\right] + \phi_{i\theta} - \left[\varepsilon_{i\theta} - \frac{\xi_i}{2}\left(R\varepsilon_k + \frac{\varepsilon_\theta}{R}\right)\right], \quad (34)$$

where $R = k_\theta^{1/2}/k^{1/2}$. One may interpret the terms on the RHS as production anisotropy, diffusive-transport, pressure temperature-gradient correlation and dissipation anisotropy. It is noted that the $P_{i\theta}$ here includes the production due to mean temperature-gradient $P^{\mathrm{T}}_{j\theta}$, the production due to mean velocity gradient $P^{U}_{i\theta}$ and the Coriolis production $C_{i\theta}$. The weak-equilibrium assumption takes the diffusive-transport as negligible, which leads to an algebraic approximation for the transport of turbulent normalized heat flux.

The budget of the various terms in Eq. (34) is evaluated by using DNS data (Nishimura and Kasagi, 1996; Kasagi and Iida, 1999; Elsamni and Kasagi, 2001). Fig. 2a and b show the budget of ξ_1-component for the non-rotating case, where the production anisotropy is the dominant source, while the pressure temperature-gradient correlation is the dominant sink. Closer to the wall, the production anisotropy and pressure temperature-gradient correlation decay rapidly, and the diffusive-transport and dissipation rate anisotropy appear as the dominant source and sink. For the rotating cases, the significant influence of rotation effects can be found by examining Fig. 2c–f, where the turbulent intensity is enhanced along pressure side, while reduced along the suction side. The production anisotropy becomes the dominant sink for the rotating case, which contrasts with the non-rotating case. The diffusive-transport is obviously enhanced by the imposed rotation effect, since it becomes the dominant source rather than sink term. The pressure temperature-gradient correlation is suppressed gradually with increasing rotation number, and becomes less important across the channel. The dissipation anisotropy is also remarkably influenced by the system rotation, since its sign is changed with different rotation numbers. Nevertheless, for the near-wall region, the diffusive-transport and dissipation anisotropy, which are related to viscous effects, are crucial.

The behavior of the individual terms in the budget equation for ξ_2-component is different with that of ξ_1-component. Since it is the wall-normal component, the pressure fluctuation dominates the near-wall behavior of budget equation instead of the viscous term. For the non-rotating case, Fig. 3a and b show that the dominant source is the pressure temperature-gradient correlation, while the production anisotropy is the sink. Closer to the wall, the production anisotropy and dissipation anisotropy become less important, while the pressure temperature-gradient correlation and diffusive-transport, which are related to the pressure fluctuation, keep balance with each other, and attain finite values on the wall. For the rotating cases, similar to ξ_1-component, the imposed rotation effects influence the budget significantly. The diffusive-transport becomes more important, while the dissipation anisotropy continues to be small across the channel. Similar to the non-rotating case, the diffusive-transport and pressure temperature-gradient correlation, which are related to the pressure fluctuation, become more important near the wall.

For all cases discussed above, the pressure temperature-gradient correlation balances the sum of the diffusive-transport plus dissipation anisotropy, which indicates that the diffusive-transport plays a crucial role in the budget of ξ_i transport equation for near-wall region. Consequently, the diffusive-transport constraint, that neglects the diffusion-transport term, is unlikely to hold there.

4.2. Modification of diffusive-transport constraint

The analysis of the budget of the ξ_i transport equation has shown that the current diffusive-transport constraint is not valid. Qiu et al. (2008) attempted to resolve this problem by representing the diffusive-transport by the sum of redistribution and dissipation anisotropy terms associated with the Reynolds stress anisotropy. For current study, this same approach will be extended to the diffusive-transport assumption associated with normalized turbulent heat flux. First, the near-wall behavior of individual term in the budget equation of ξ_i is analyzed. Based on that analysis, the near-wall correction for diffusive-transport constraint is proposed.

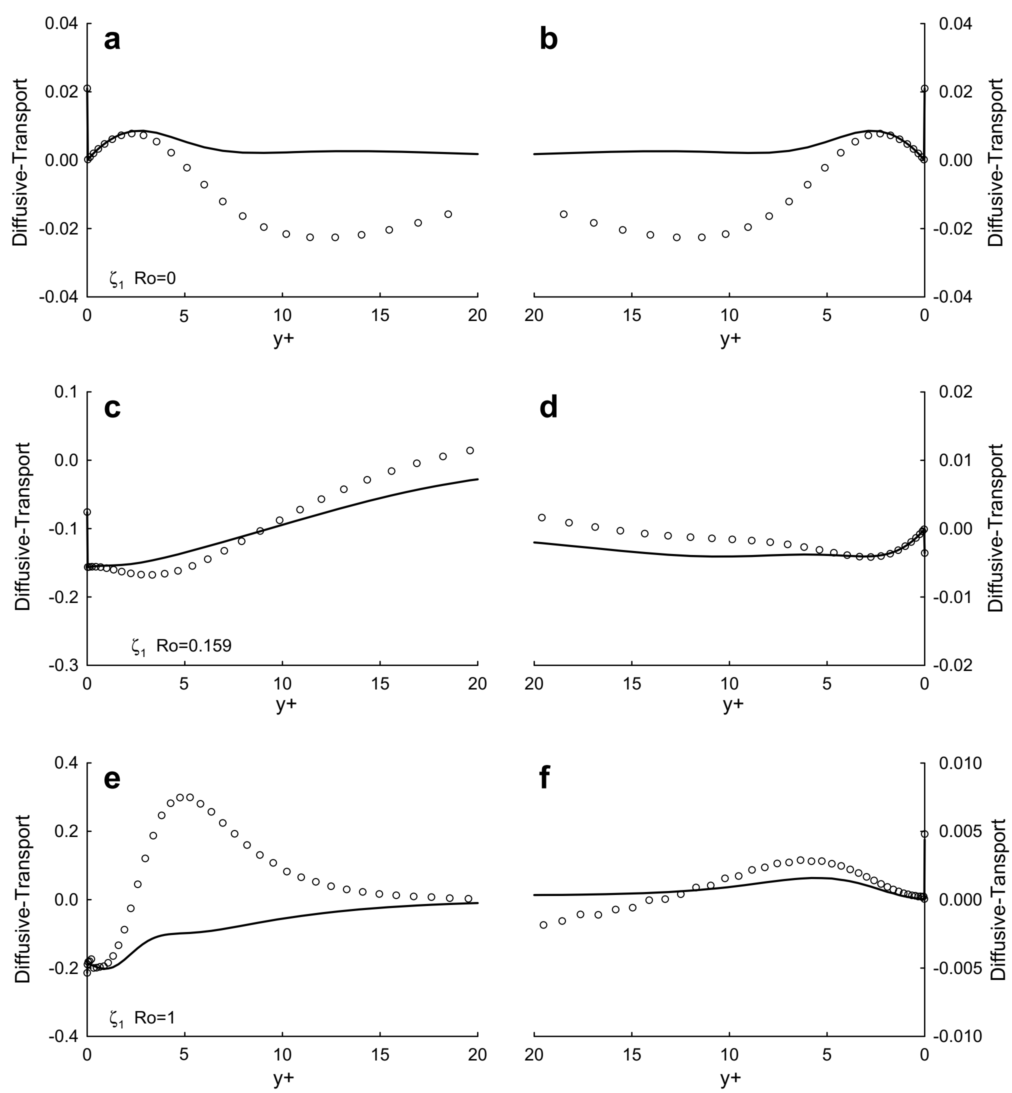

Fig. 4. Validation of present near-wall correction of diffusive-transport constraint for ξ_1 LHS: pressure side, RHS: suction side. The present constraint (solid line) compared to DNS (circle).

4.2.1. Near-wall behavior of normalized heat flux equation

To analyze the near-wall behavior of individual terms in the budget equation, the pressure, velocity and temperature fluctuations (Wikström et al., 2000; So et al., 2004) are expanded in a Taylor series in the vicinity of the wall as follows:

$$p = a_p + b_p y + c_p y^2 + \cdots, \tag{35a}$$

$$u = a_i + b_i y + c_i y^2 + \cdots, \tag{35b}$$

$$\theta = a_\theta + b_\theta y + c_\theta y^2 + \cdots, \tag{35c}$$

where $a_i = b_2 = a_\theta = 0$ (no-slip boundary condition, continuity and constant wall temperature).

The expansions of $\overline{u\theta}$, $\overline{v\theta}$ and k_θ then become

$$\overline{u\theta} = \overline{b_\theta b_1} y^2 + (\overline{b_\theta c_1} + \overline{c_\theta b_1}) y^3 + \cdots, \tag{36a}$$

$$\overline{u\theta} = \overline{b_\theta c_2} y^2 + (\overline{b_\theta d_2} + \overline{c_\theta c_2}) + \cdots, \tag{36b}$$

$$k_\theta = \frac{1}{2}\overline{b_\theta^2} y^2 + \overline{b_\theta c_\theta} y^3 + \cdots. \tag{36c}$$

The expanded budget terms in Eq. (34) are listed in Table 1, where only terms $\mathscr{O}(y^0)$ are listed, and budget terms of second order and higher are omitted. For ξ_1-component, the diffusive-transport and dissipation anisotropy are of lower order compared to the production anisotropy and pressure temperature-gradient correlation, which indicates that they are more important in the near-wall region. The production anisotropy and pressure temperature-gradient correlation decay rapidly in the near-wall region; while the diffusive-transport balances the dissipation anisotropy up to the wall. For ξ_2-component, Table 1 shows that the diffusive-transport and pressure temperature-gradient correlation are the major contributors near the wall, while the production anisotropy and dissipation anisotropy decay faster and eventually vanish on the wall.

4.2.2. Near-wall correction of diffusive-transport constraint

The above analysis indicates that for the ξ_1- and ξ_2-component, the fact that diffusive-transport balances the sum of pressure temperature-gradient correlation plus dissipation anisotropy suggests an alternative form for diffusive-transport constraint. One can use the sum of the pressure temperature-gradient correlation plus dissipation anisotropy to represent the diffusive-transport in the near-wall region. Thus, based on the above analyses, an alternative form for the diffusive-transport constraint can be given by

$$\mathscr{D}_{i\theta} - \frac{\xi_i}{2}\left(R\mathscr{D}_k + \frac{\mathscr{D}_\theta}{R}\right) = -\left\{\phi_{i\theta} - \left[\varepsilon_{i\theta} - \frac{\xi_i}{2}\left(R\varepsilon_k + \frac{\varepsilon_\theta}{R}\right)\right]\right\} f_t, \tag{37}$$

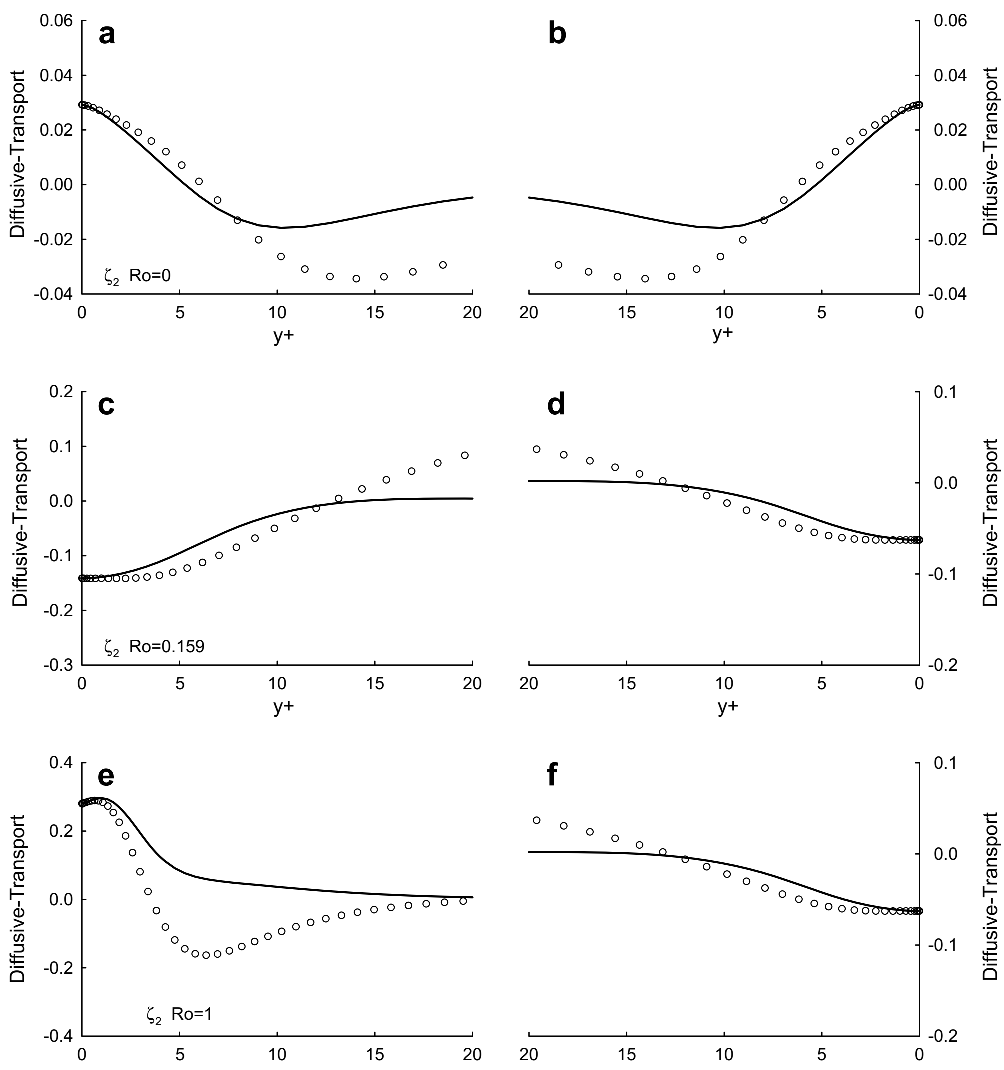

Fig. 5. Validation of present near-wall correction of diffusive-transport constraint for ξ_2 LHS: pressure side, RHS: suction side. The present constraint (solid line) compared to DNS (circle).

where f_t is a model function, and for the present case, the following general form can be used

$$f_t = 1 - \left[1 - \exp\left(\frac{y^+}{\lambda}\right)\right]^2, \tag{38}$$

where $y^+ = yu_\tau/v$ and u_τ is the friction velocity. The Prandtl number *Pr* and Reynolds number *Re* have a significant influence on the scalar field (Kim and Moin, 1987; Kawamura et al.,1998, 1999). So and Speziale (1999) suggested any near-wall models should reflect a *Pr* dependence; otherwise, they would not be able to replicate the thermal asymptotes correctly as a wall is approached. The above arguments imply that the proposed diffusive-transport constraint should be made parametric of *Pr*, which means that the λ should have a *Pr* dependence; however, in the current study, a constant $\lambda = 6$ value was chosen, but its universal validity should be a topic for future study. As Eq. (38) shows, f_t approaches unity in the vicinity of the wall, and vanishes away from the wall. Consequently, one can restrict the effects of pressure temperature-gradient correlation and dissipation anisotropy within the near-wall region.

From the above asymptotic behavior and budget analysis, the near-wall diffusion and transport of ξ_i are mainly due to pressure transport and viscous diffusion effects, and both of which must be properly approximated. For the ξ_1-component, the pressure fluctuation contributions on both sides of Eq. (37) are negligible, which means that the viscous diffusion part of the LHS balances the viscous dissipation anisotropy part of the RHS. For the ξ_2-component, the viscous parts of both sides of Eq. (37) are negligible, which leaves the pressure transport part of the LHS to balance the pressure fluctuation contribution on the RHS.

4.2.3. Evaluation of proposed constraint

By adopting the present diffusive-transport constraint, Eq. (34) can now be written as

$$0 = \left[P_{i\theta} - \frac{\xi_i}{2}\left(RP_k + \frac{P_\theta}{R}\right)\right] + \left\{\phi_{i-\theta} - \left[\varepsilon_{i_\theta} - \frac{\xi_i}{2}\left(R\xi_k + \frac{\varepsilon_\theta}{R}\right)\right]\right\}(1 - f_t). \tag{39}$$

This equation indicates that the diffusive-transport constraints need to be incorporated in a manner so that both the pressure temperature-gradient correlation and dissipation anisotropy disappear near the wall. It should be noted that the validity of this form is unaffected by the system rotation since the balance between the pressure temperature-gradient correlation and pressure transport and that between the viscous diffusion and dissipation anisotropy persist regardless of the rotation number as has been shown in Figs. 2 and 3.

An *a priori* test is performed to evaluate the present diffusive-transport constraint given in Eq. (37). Both sides are computed using the DNS data and compared with each other. For the ξ_1-component, Fig. 4 shows that the newly proposed diffusive-transport constraint gives fairly good agreement with DNS data for $y^+ \leqslant 5$ for both the non-rotating and rotating cases, and which is an improvement compared with the original proposal. For the ξ_2-component, Fig. 5 shows that the present diffusive-transport constraint can also give good agreement for $y^+ \leqslant 10$ compared to the DNS data.

It should be noted that when compared with the Reynolds stress case (Qiu et al., 2008), where the extended constraint gave excellent agreement with the DNS data for all Reynolds stress components, the extended constraint developed in the present study yields some noticeable discrepancies with the DNS data in some cases. In both the Reynolds stress and heat flux cases, the production terms are assumed to be negligible. This assumption is well supported in the case of the Reynolds stresses, of which, the production becomes negligible in the vicinity of the wall. However, for the heat flux cases, the production remains small but non-negligible for $y^+ \leqslant 10$ in some cases (see Figs. 2 and 3). Nevertheless, the above analysis indicates that the present alternative diffusive-transport constraint has the potential to improve the AHFM, once accurate models for pressure temperature-gradient correlation and dissipation anisotropy in Eq. (34) are provided.

5. Conclusion

This study has focused on the validity and extensions of the weak-equilibrium condition in non-inertial frames. The weak-equilibrium condition, which consists of an advection assumption and a diffusive-transport constraint, is the basis to derive the algebraic heat flux model from the differential transport model. The frame-invariant concept is invoked in this study to extend the original advection assumption for flows associated with rotation and curvature effects. Moreover, the frame-invariant form of AHFM is derived by using the extended weak-equilibrium condition. It is also shown that the transport equation of normalized heat flux can be written in a Euclidean invariant way by introducing the Jaumann–Noll derivative.

A budget analysis of the various terms in the exact transport equation for ξ_i shows that the diffusive-transport is crucial in the near-wall region. An asymptotic analysis of the near-wall behavior shows that, the diffusive-transport keeps balance with the sum of pressure temperature-gradient correlation plus dissipation anisotropy in the vicinity of the wall, while production anisotropy is small. An alternative form of the diffusive-transport constraint is proposed and evaluated using DNS data. The analysis shows that the newly proposed alternative constraint has the potential to improve the predictive ability of the resultant AHFM.

Acknowledgement

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